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Areas of planar Brownian curves

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Abstract. We address the problem of the algebraic area enclosed by a Brownian curve in two dimensions, recently reconsidered by Khandekar and Wiegel. We recall the principal results actually first obtained by Paul Lévy in 1950. Another derivation by functional integrals is given for several probability distributions: that of the area of a Brownian ring, the area between an open Brownian arc and its chord and, finally, the area swept with respect to the mean position of the Brownian path, a result which seems to be new.

1. Introduction

In a recent paper Khandekar and Wiegel (1988) (hereafter referred to as KW) obtained the distribution function of the algebraic area of a closed Brownian curve in two dimensions. They were elaborating upon a study by Brereton and Butler (1987) of the areas spanned by discrete Gaussian walks in two dimensions. Let us fix the notation and take the continuum Brownian probability weight as

$$P\{\boldsymbol{r}\} = \exp\left(-\frac{1}{2}\int_0^S \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\right)^2 \mathrm{d}s\right) \tag{1.1}$$

where $\boldsymbol{r}(s) = (x(s), y(s))$ is the configuration in $\mathbb{R}^2$, depending on abscissa $0 \leqslant s \leqslant S$, S being the 'length' of the path. The average end-to-end distance is then, with respect to the above weight,

$$R^2 = \langle[\boldsymbol{r}(S) - \boldsymbol{r}(0)]^2\rangle = 2S. \tag{1.2}$$

Then, considering now closed Brownian rings of size S, the probability to enclose an *algebraic* area in two dimensions (figure 1)

$$\mathscr{A}\{\boldsymbol{r}\} = \frac{1}{2}\int_0^S \left(x\frac{\mathrm{d}y}{\mathrm{d}s} - y\frac{\mathrm{d}x}{\mathrm{d}s}\right) \mathrm{d}s \tag{1.3}$$

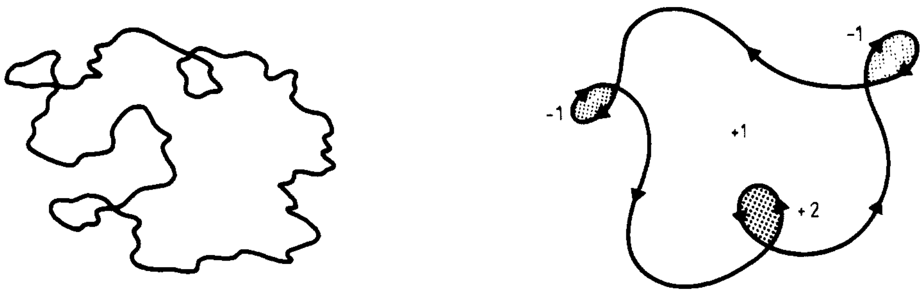

Figure 1. Algebraic area enclosed by a planar Brownian ring. The hatched exterior loops contribute negatively while the hatched interior loop contributes twice its arithmetic area.

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is

$$P(A) = \langle \delta(A - \mathscr{A}\{r\}) \rangle_{\text{ring}} \tag{1.4}$$

where the subscript represents the closure constraint. Its expression is found to be (KW)

$$P(A) = (\pi/2S)(\cosh \pi A/S)^{-2} \tag{1.5}$$

and is properly normalised as

$$\int_{-\infty}^{+\infty} P(A)\, \mathrm{d}A = 1$$

since $\int_0^{+\infty} (\cosh x)^{-2}\, \mathrm{d}x = 1$. In Khandekar and Wiegel's notation we have $2S \equiv Nl^2$ and the above probability is $P(A, N) = (\pi/Nl^2)(\cosh 2\pi A/Nl^2)^{-2}$ (a factor 2π being missing in (KW, 24) for a proper normalisation).

KW obtain (1.5) by a functional integral technique starting from the continuum model (1.1) and calculating the generating function (Fourier transform) of (1.5). The latter is given by a path integral similar to that of a quantum mechanical particle in a magnetic field.

Brereton and Butler (1987) studied the same problem for discrete Gaussian random walks by an expansion in normal-mode coordinates. They did not however obtain (1.5) in a closed form, but gave instead a numerical curve for $P(A)$. They also studied the area swept by an open Gaussian walk with respect to a fixed point (figure 2(b)), as well as the average end-to-end distance with a fixed area.

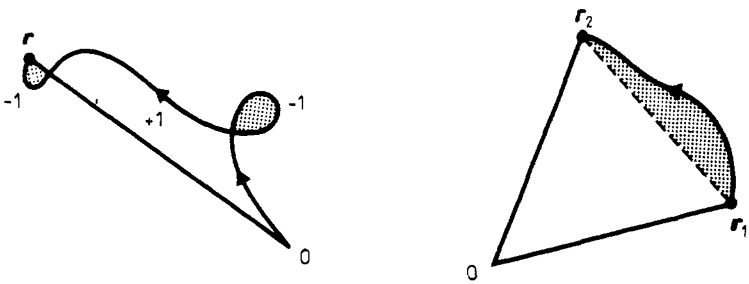

Figure 2. Areas enclosed by an open Brownian path (a) between the arc and the chord, and (b) between the arc and the end-to-end triangle. In (b) the area counted includes the shaded and triangle areas.

The purposes of this paper are several. First of all, we want to comment that studies about the area described by a planar Brownian motion were initiated by Lévy as early as 1940, who first obtained the distribution (1.5) (Lévy 1950, 1951). He made use of the expansion by Wiener (1924) for a random Brownian variable in stochastic Fourier series (normal modes). He considered also (Lévy 1950) the algebraic area confined between a Brownian arc starting at $\boldsymbol{r}(0) = \boldsymbol{0}$ and ending at $\boldsymbol{r}(S) = \boldsymbol{r}$, and its *chord* (figure 2(a)). The characteristic function is then (Lévy 1951)

$$\mathbb{E}[\exp(\mathrm{i}g\mathscr{A}) \,|\, \boldsymbol{r}(S) - \boldsymbol{r}(0) = \boldsymbol{r}] = \frac{u}{\sinh u} \exp[(1 - u \coth u) r^2/2S] \tag{1.6}$$

where

$$u = gS/2. \tag{1.7}$$

$\mathbb{E}$ above is defined as an expectation value, *conditional* on the extremity $\boldsymbol{r}(S)$ being at distance $\boldsymbol{r}$ from the origin of the path (a similar formula appears in KW). It can be written in terms of the generating function for a free open Brownian motion

$$\mathscr{Z}(g, \boldsymbol{r}) \equiv \langle \exp(\mathrm{i}g\mathscr{A})\delta^2(\boldsymbol{r}(S) - \boldsymbol{r}(0) - \boldsymbol{r})\rangle \tag{1.8}$$

where the average is taken with weight (1.1). One has

$$\mathbb{E}(g, r) = \mathscr{Z}(g, \boldsymbol{r})/\mathscr{Z}(0, \boldsymbol{r}) \tag{1.9}$$

where

$$\mathscr{Z}(g = 0, \boldsymbol{r}) = (2\pi S)^{-1} \exp(-r^2/2S) \tag{1.10}$$

is the end-to-end distance probability distribution of a 2D Brownian motion. Hence Lévy's result equivalently is

$$\mathscr{Z}(g, \boldsymbol{r}) = \frac{u}{2\pi S \sinh u} \exp(-u \coth u\, r^2/2S). \tag{1.11}$$

Taking the zero-distance limit $r = 0$ of (1.6), one gets the characteristic function for a planar closed Brownian ring (Lévy 1950, 1951)

$$\mathbb{E}(g, 0) = u/\sinh u \tag{1.12}$$

which by Fourier transforming

$$P(A) = \int_{\mathbb{R}} \exp(-\mathrm{i}gA)\mathbb{E}(g, 0)\, \mathrm{d}g/2\pi \tag{1.13}$$

gives the announced result (1.5).

The total characteristic function of the area between an arc of the Brownian curve and its chord is obtained by integrating over $\boldsymbol{r}$ in $\mathbb{R}^2$ (Lévy 1950)

$$\mathscr{Z}_{\text{chord}}(g) = \int_{\mathbb{R}^2} \mathscr{Z}(g, \boldsymbol{r})\, \mathrm{d}^2 r = \frac{1}{\cosh u}. \tag{1.14}$$

Fourier transforming with respect to g yields

$$P_{\text{chord}}(A) = \frac{1}{S \cosh(\pi A/S)}. \tag{1.15}$$

Note that Fourier inverting with respect to g the full joint distribution (1.6) in g and r seems difficult.

In this paper, we want to give a short review of the methods used in the literature to determine these characteristic functions and probabilities, and give also an alternative derivation by standard functional integrals, which is simpler than that of KW. We should first stress that the mode expansion papers by Lévy (1950, 1951) are quite elegant. Another simple derivation of the characteristic function (1.6) has been given by probabilistic methods (Yor 1980), using more recent techniques for the Brownian motion, i.e. the absolute continuity between the laws of an Ornstein–Uhlenbeck process and Brownian motion, on the interval $[0, S]$. Extensions of Lévy's stochastic area formula were also obtained through its relation to Legendre polynomials (Biane and Yor 1986, 1987), in Lévy's tradition of deriving a wealth of explicit formulae.

Let us discuss now the functional integral technique of Khandekar and Wiegel. They evaluate quite classically the generating function (1.8) by Gaussian integration. However, they use a saddle point trajectory, which is determined by Euler-Lagrange equations, about which they calculate the quadratic fluctuations. This method is indeed standard for evaluating the Feynman-Kac integrals for harmonic oscillators. But the determination in KW of this saddle point trajectory is 'remarkably tedious', as noted by the authors themselves, and leads to complicated calculations. In the next section we give a direct functional integral calculation of (1.8), which avoids these difficulties. The method is generic and could also be useful to mathematicians, for a comparison with the probabilistic methods. For completeness, we also derive in § 3 by the same method the generating function and probability distribution of the area swept by a Brownian path with respect to its *centre of gravity* (i.e. its mean position).

Before proceeding to the calculation, let us add a last comment about Brereton and Butler's work (1987, hereafter referred to as BB). They obtain for the normalised generating function of the area ((3.18) and (4.1) of their work) of a discrete closed Gaussian walk of N steps of mean length l:

$$\mathbb{E}(g,0) \equiv Z(g,N) = \prod_{n=1}^{N-1} (1+g^2 l^4 \beta_n''^2)^{-1/2} \tag{1.16}$$

where

$$\beta_n'' = \tfrac{1}{4}\cot(\pi n/N). \tag{1.17}$$

We have added an exponent $\frac{1}{2}$ in (1.16) which is apparently missing in their equation (3.18). BB do not evaluate (1.16) in a closed form but resort on numerical approximations. One can actually recover Lévy's Brownian result (1.12) from (1.16) and (1.17) in the long walk, i.e. continuum limit $N \to \infty$. For this it is sufficient to use the periodicity of cot to get

$$Z(g,N) \approx \prod_{n=1}^{N/2} (1+g^2 l^4 \beta_n''^2)^{-1} \tag{1.18}$$

and note that then the dominant terms will come from the neighbourhood of the origin in n, where $\beta_n'' \approx N/4\pi n$. Hence

$$Z(g,N) \approx \prod_{n=1}^{\infty} \left[1+\left(\frac{gNl^2}{4\pi n}\right)^2\right]^{-1} \tag{1.19}$$

where the product can be completed up to infinity. Using the well known product expansion $\sinh u/u = \Pi_{n\in\mathbb{N}^*}(1+u^2/\pi^2 n^2)$, we get asymptotically

$$Z(g,N) \approx \frac{u}{\sinh u} \qquad u = gNl^2/4$$

which is nothing other than (1.12) for $u = gS/2$ (1.7) and $S \equiv Nl^2/2$, as expected. So result (3.18) of Brereton and Butler (1987), slightly amended, also leads to Lévy's formula, as it must.

Let us now present an alternative and simple derivation by standard functional methods.

2. Generating functions and probability distributions

2.1. Functional integrals

In order to compute results for all geometrical cases, we introduce the area-generating function for Brownian motion with fixed extremities (figure 2) $\boldsymbol{r}(0)=\boldsymbol{r}_1$, $\boldsymbol{r}(S)=\boldsymbol{r}_2$, $\boldsymbol{r}=\boldsymbol{r}_2-\boldsymbol{r}_1$:

$$\mathscr{Z}(g,\boldsymbol{r}_1,\boldsymbol{r}_2)\equiv\int \mathrm{d}\{\boldsymbol{r}\}\exp\left(-\tfrac{1}{2}\int_0^S \dot{\boldsymbol{r}}^2(s)\,\mathrm{d}s+\mathrm{i}g\mathscr{A}\{\boldsymbol{r}\}\right)\delta^2(\boldsymbol{r}(0)-\boldsymbol{r}_1)\delta^2(\boldsymbol{r}(S)-\boldsymbol{r}_2)/\mathscr{Z}_0 \tag{2.1a}$$

where the normalisation factor is the partition function of a free Brownian motion, with the origin fixed:

$$\mathscr{Z}_0=\int \mathrm{d}\{\boldsymbol{r}\}\exp\left(-\tfrac{1}{2}\int \dot{\boldsymbol{r}}^2\,\mathrm{d}s\right)\delta^2(\boldsymbol{r}(0)) \tag{2.1b}$$

such that

$$\int \mathscr{Z}(g=0,\boldsymbol{r}_1,\boldsymbol{r}_2)\,\mathrm{d}^2r\equiv 1. \tag{2.1c}$$

From now on, we *include* the normalisation factor $\mathscr{Z}_0^{-1}$ in the *definition* of the functional measure.

The area $\mathscr{A}$ (1.3) is essentially a quadratic form of $x(s)$, $y(s)$. It is most convenient to rewrite it in terms of the derivative $\dot{\boldsymbol{r}}(s)=(\mathrm{d}\boldsymbol{r}/\mathrm{d}s)(s)$, by using in (1.3) the trivial identity

$$x(s)=\int_0^S \mathrm{d}s'\theta(s-s')\dot{x}(s')+x(0) \tag{2.2}$$

and a similar equation for $y(s)$. The vector product is also conveniently represented by use of the 2×2 antisymmetric tensor:

$$\boldsymbol{\varepsilon}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

such that

$$xy'-yx'=\boldsymbol{r}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{r}'. \tag{2.3}$$

The area (1.3) is therefore

$$\mathscr{A}\{\boldsymbol{r}\}=-\tfrac{1}{2}\int_0^S \mathrm{d}s\int_0^S \mathrm{d}s'\dot{\boldsymbol{r}}(s)\cdot\mathbb{B}(s,s')\cdot\dot{\boldsymbol{r}}(s')+\tfrac{1}{2}\boldsymbol{r}_1\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{r}_2 \tag{2.4}$$

where $\mathbb{B}(s,s')$ is the *symmetric* operator:

$$\mathbb{B}(s,s')=\tfrac{1}{2}[\theta(s-s')-\theta(s'-s)]\boldsymbol{\varepsilon}. \tag{2.5}$$

Notice that the first term of (2.4) is the algebraic area enclosed between the arc of the curve and its chord, while the second is just the area of the triangle spanned by $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ with respect to the origin (figure 2). It is essential to symmetrise $\mathbb{B}$ under the exchange of s, s' and of the space indices in order to get a symmetric Gaussian integral later.

The integration variables in (2.1) are changed to $\dot{\boldsymbol{r}}(s)$ by the use of $\mathrm{d}\{\boldsymbol{r}(s)\} = \mathrm{d}^2 r(0)\, \mathrm{d}\{\dot{\boldsymbol{r}}(s)\}$, and (2.1) becomes

$$\mathscr{Z}(g, \boldsymbol{r}_1, \boldsymbol{r}_2) = \exp(\mathrm{i}g\tfrac{1}{2}\boldsymbol{r}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{r}_2) \int \mathrm{d}\{\dot{\boldsymbol{r}}(s)\}$$
$$\times \exp\left(-\tfrac{1}{2}\int \mathrm{d}s\, \mathrm{d}s' \dot{\boldsymbol{r}}(s) \cdot \mathbb{A}(s, s') \cdot \dot{\boldsymbol{r}}(s')\right)\delta^2\left(\boldsymbol{r} - \int_0^S \dot{\boldsymbol{r}}(s)\, \mathrm{d}s\right) \tag{2.6}$$

where $\mathbb{A}$ is the symmetric operator:

$$\mathbb{A}(s, s') = \delta(s - s')\mathbf{1} + \mathrm{i}g\tfrac{1}{2}[\theta(s - s') - \theta(s' - s)]\boldsymbol{\varepsilon} \tag{2.7}$$

and $\mathbf{1}$ is the unit 2×2 matrix. The constraint on the total displacement is Fourier transformed and (2.6) becomes

$$\mathscr{Z}(g, \boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \exp(\tfrac{1}{2}\mathrm{i}g\boldsymbol{r}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{r}_2)\mathscr{Z}(g, \boldsymbol{r}) \tag{2.8a}$$

$$\mathscr{Z}(g, \boldsymbol{r}) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})\tilde{\mathscr{Z}}(g, \boldsymbol{k}) \tag{2.8b}$$

$$\tilde{\mathscr{Z}}(g, \boldsymbol{k}) = \int \mathrm{d}\{\dot{\boldsymbol{r}}(s)\} \exp\left(-\tfrac{1}{2}\int \dot{\boldsymbol{r}} \cdot \mathbb{A} \cdot \dot{\boldsymbol{r}} + \mathrm{i}\int \boldsymbol{k} \cdot \dot{\boldsymbol{r}}(s)\, \mathrm{d}s\right). \tag{2.9}$$

Everything has been reduced to calculating Gaussian integrals. We recall the discrete formula

$$\int \prod_{i=1}^{n} \mathrm{d}x_i \exp(-\tfrac{1}{2}X \cdot A \cdot X + H \cdot X) = (2\pi)^{n/2}(\det A)^{-1/2} \exp(\tfrac{1}{2}H \cdot A^{-1} \cdot H) \tag{2.10}$$

valid for vectors $X = (x_1 \ldots, x_n)$ and H of $\mathbb{R}^n$ with $H \cdot X = \Sigma_i h_i x_i$, and A a symmetric positive-definite matrix. In the continuum limit this gives the Gaussian functional integral (2.9):

$$\tilde{\mathscr{Z}}(g, \boldsymbol{k}) = (\det \mathbb{A})^{-1/2} \exp\left(-\tfrac{1}{2}k^2 \int_0^S \int_0^S A'(s, s')\, \mathrm{d}s\, \mathrm{d}s'\right) \tag{2.11}$$

or in direct space

$$\mathscr{Z}(g, \boldsymbol{r}) = \left(2\pi \int_0^S \int_0^S A'(s, s')\, \mathrm{d}s\, \mathrm{d}s'\right)^{-1} (\det \mathbb{A})^{-1/2}$$
$$\times \exp\left[-\tfrac{1}{2}r^2\left(\int_0^S \int_0^S A'(s, s')\, \mathrm{d}s\, \mathrm{d}s'\right)^{-1}\right]. \tag{2.12}$$

In these formulae A' is the space-symmetric part of the inverse $\mathbb{A}^{-1}$ of the operator $\mathbb{A}$, such that

$$\mathbb{A}^{-1}(s, s') = A'(s, s')\mathbf{1} + B'(s, s')\boldsymbol{\varepsilon}. \tag{2.13}$$

Let us recall that $\mathscr{Z}(g, \boldsymbol{r})$ is the generating function of the area $\mathscr{A}_{\text{chord}}$ enclosed between the chord and the arc, while $\mathscr{Z}(g, \boldsymbol{r}_1, \boldsymbol{r}_2)$ is that of the total area spanned between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ with respect to a fixed origin O (figure 2). Because of the normalisation (2.1c), we have explicitly

$$\mathscr{Z}_{\text{chord}}(g, \boldsymbol{r}) = \langle \exp(\mathrm{i}g\mathscr{A}_{\text{chord}})\delta^2(\boldsymbol{r}(S) - \boldsymbol{r}(0) - \boldsymbol{r})\rangle. \tag{2.14}$$

The generating function of the arc–chord area, *without* the end-to-end distance constraints, is accordingly

$$\begin{aligned}\mathscr{Z}_{\text{chord}}(g) &\equiv \langle \exp(\mathrm{i}g\mathscr{A}_{\text{chord}})\rangle \\ &= \int_{\mathbb{R}^2} \mathscr{Z}(g, \boldsymbol{r})\,\mathrm{d}^2 r = \tilde{\mathscr{Z}}(g, \boldsymbol{k}=0) \\ &= (\det \mathbb{A})^{-1/2}.\end{aligned} \tag{2.15}$$

We have obtained these functions in closed form (2.12) and it now remains to evaluate explicitly the determinant det $\mathbb{A}$ and the inverse $\mathbb{A}^{-1}$ (2.13) of operator $\mathbb{A}$.

2.2. *Inversion of* $\mathbb{A}$

The operator $\mathbb{A}$ having been given in (2.7), and $\mathbb{A}^{-1}$ in (2.13), the equation $\mathbb{A}\mathbb{A}^{-1} = \mathbb{1}$ is explicitly written in the $\mathbf{1}$, $\boldsymbol{\varepsilon}$ matricial basis

$$\begin{aligned}&A'(s, s') - \tfrac{1}{2}\mathrm{i}g\left(\int_0^s \mathrm{d}s'' - \int_s^S \mathrm{d}s''\right) B'(s'', s') = \delta(s - s') \\ &B'(s, s') + \tfrac{1}{2}\mathrm{i}g\left(\int_0^s \mathrm{d}s'' - \int_s^S \mathrm{d}s''\right) A'(s'', s') = 0.\end{aligned} \tag{2.16}$$

We integrate the kernels on the right variable s' and define

$$\begin{aligned}X(s) &\equiv \int_0^S [A'(s, s') + \mathrm{i}B'(s, s')]\,\mathrm{d}s' \\ Y(s) &\equiv \int_0^S [A'(s, s') - \mathrm{i}B'(s, s')]\,\mathrm{d}s'.\end{aligned} \tag{2.17}$$

Equations (2.16) then combine into

$$\begin{aligned}&X(s) - \tfrac{1}{2}g\left(\int_0^s \mathrm{d}s'' - \int_s^S \mathrm{d}s''\right) X(s'') = 1 \\ &Y(s) + \tfrac{1}{2}g\left(\int_0^s \mathrm{d}s'' - \int_s^S \mathrm{d}s''\right) Y(s'') = 1.\end{aligned} \tag{2.18}$$

Differentiating both sides with respect to s gives

$$\mathrm{d}X/\mathrm{d}s = gX \qquad \mathrm{d}Y/\mathrm{d}s = -gY$$

which are trivially integrated into

$$X(s) = \lambda\,\mathrm{e}^{gs} \qquad Y(s) = \mu\,\mathrm{e}^{-gs}. \tag{2.19}$$

The constants λ, μ are determined by plugging the forms (2.19) into integral equations (2.18):

$$\lambda = \mathrm{e}^{-gS/2}(\cosh gS/2)^{-1} \qquad \mu = \mathrm{e}^{gS/2}(\cosh gS/2)^{-1}. \tag{2.20}$$

We therefore find, owing to (2.17),

$$\int_0^S A'(s, s')\,\mathrm{d}s' = \tfrac{1}{2}[X(s) + Y(s)] = \frac{\cosh g(s - S/2)}{\cosh gS/2}.$$

The double integral, appearing in (2.12), is finally

$$\int_0^S \int_0^S A'(s, s')\,\mathrm{d}s\,\mathrm{d}s' = \frac{2}{g}\tanh(gS/2). \qquad (2.21)$$

Notice that one finds similarly $\int_0^S \int_0^S B'(s, s')\,\mathrm{d}s\,\mathrm{d}s' = 0$, as expected from the antisymmetry of B' in (2.13).

2.3. Determinant of $\mathbb{A}$

We formally write (2.7) as $\mathbb{A} = \mathbb{1} + \mathrm{i}g\mathbb{B}$ and

$$\det \mathbb{A} = \exp \operatorname{Tr} \ln(\mathbb{1} + \mathrm{i}g\mathbb{B}) = \exp \sum_{n\geq 1} (-1)^{n+1}(1/n)(\mathrm{i}g)^n \operatorname{Tr} \mathbb{B}^n.$$

Owing to the trace properties

$$\operatorname{Tr} \boldsymbol{\varepsilon} = 0 \qquad \boldsymbol{\varepsilon}^2 = -\mathbf{1} \qquad \operatorname{Tr} \mathbf{1} = 2$$

the above summation is reduced to even integers $n = 2n'$ only and

$$\ln \det \mathbb{A} = -\sum_{n'=1}^{\infty} \frac{1}{n'}\left(\frac{g}{2}\right)^{2n'} \operatorname{Tr}[(\theta^+ - \theta^-)^{2n'}] \qquad (2.22)$$

where the meaning of the symbolic operator $(\theta^+ - \theta^-)^{2n'}$ is clear from (2.5):

$$\operatorname{Tr}[(\theta^+ - \theta^-)^{2n'}] = \int_0^S \prod_{i=1}^{2n'} \mathrm{d}s_i \prod_{i=1}^{2n'} h(s_i - s_{i+1}) \qquad (2.23a)$$

with $s_{2n'+1} \equiv s_1$ and

$$h(s) \equiv \theta(s) - \theta(-s). \qquad (2.23b)$$

The easiest way to compute (2.23) is to diagonalise (2.23b) as an integral operator with an eigenvalue equation

$$\int_0^S h(s - s')g(s')\,\mathrm{d}s' = \lambda g(s).$$

The solutions are trivially

$$g_m(s) = \exp(2s/\lambda_m)$$

$$\exp(2S/\lambda_m) = -1 \qquad \lambda_m = 2S/\mathrm{i}\pi m \qquad m \in 2\mathbb{Z} + 1.$$

Hence we have immediately

$$\operatorname{Tr}[(\theta^+ - \theta^-)^{2n'}] = \sum_{m\in 2\mathbb{Z}+1} \lambda_m^{2n'}$$

$$= \sum_{m\in 2\mathbb{Z}+1} (-1)^{n'}\left(\frac{2S}{\pi m}\right)^{2n'}. \qquad (2.24a)$$

Notice that one can also use here a Fourier mode decomposition of $h(s)$ over the interval $[-S, S]$

$$h(s) = \sum_{m\in\mathbb{Z}} \tilde{h}_m \exp(\pi\mathrm{i}ms/S) \qquad \tilde{h}_m = \frac{1}{\mathrm{i}\pi m}[(-1)^m - 1] = \frac{-2}{\mathrm{i}\pi m}\,\delta_{m,2\mathbb{Z}+1}$$

such that

$$\operatorname{Tr}[(\theta^+ - \theta^-)^{2n'}] = \sum_{m\in 2\mathbb{Z}+1} S^{2n'}\tilde{h}_m^{2n'}. \qquad (2.24b)$$

We may remark at this stage that this Fourier mode decomposition is the analogue of the mode expansion of the Brownian motion, as used by Lévy (1950). However, one should note that the convolution integral (2.23) is performed only on the *half* period $[0, S]$, and in general the Fourier method does not diagonalise it. Here, the fact that only *odd* modes appear in $\tilde{h}_m$ make it useful. The general method is really the diagonalisation of the integral kernel $h(s-s')$, as we shall see in § 3 when considering the area swept with respect to the centre of gravity.

Plugging (2.24) into (2.22) and performing back the summation over $n' \in \mathbb{N}$ gives

$$\ln \det \mathbb{A} = \sum_{m \in 2\mathbb{Z}+1} \ln\left(1+\frac{g^2S^2}{\pi^2 m^2}\right).$$

From the well known identity

$$\frac{\sinh z}{z} = \prod_{m \in \mathbb{N}^*} \left(1+\frac{z^2}{\pi^2 m^2}\right) \tag{2.25}$$

one can deduce the product over odd numbers

$$\left(\cosh\frac{z}{2}\right)^2 = \prod_{m \in 2\mathbb{Z}+1} \left(1+\frac{z^2}{\pi^2 m^2}\right). \tag{2.26}$$

Hence we finally get the very simple result

$$\det \mathbb{A} = \cosh^2 gS/2. \tag{2.27}$$

2.4. Generating functions

The basic generating function (2.14) for the area between the Brownian arc and its chord, for a fixed span $\boldsymbol{r}$, (figure 2(a)) has now the explicit form (use (2.21) and (2.27) in (2.12))

$$\mathscr{Z}(g, \boldsymbol{r}) = \frac{g}{4\pi} \frac{1}{\sinh(gS/2)} \exp[-gr^2/4 \tanh(gS/2)]. \tag{2.28}$$

From this generating function we deduce those of all other geometric situations. First, the case of the area of a *closed* Brownian curve (figure 1) is obtained by letting $\boldsymbol{r} \to \boldsymbol{0}$ in (2.28) to get

$$\begin{aligned}\mathscr{Z}(g) \equiv \mathscr{Z}(g, \boldsymbol{r}=\boldsymbol{0}) &= \langle \exp(\mathrm{i}g\mathscr{A})\delta^2(\boldsymbol{r}(S)-\boldsymbol{r}(0))\rangle \\ &= \frac{g}{4\pi \sinh gS/2}.\end{aligned} \tag{2.29}$$

Second, when one relaxes the constraint of a fixed distance $\boldsymbol{r}$ between the extremities one gets the full arc–chord generating function (2.15) (use (2.27)):

$$\mathscr{Z}_{\text{chord}}(g) = (\cosh gS/2)^{-1} \tag{2.30}$$

which can also be obtained by integrating (2.28) over $\boldsymbol{r}$.

Recall that, if the area of the Brownian arc is measured with respect to an external reference point O (figure 2(b)), the generating function is given by (2.8a). One can then consider a last interesting geometrical case. One fixes the origin of the curve at position r_1 with respect to O, and lets the end position move. Then the total generating function of the area swept by the random path with respect to O (figure 2(*b*)) is

$$\mathscr{Z}_{\text{path}}(g, \boldsymbol{r}_1) = \int \mathrm{d}^2 r \exp(\mathrm{i} g \boldsymbol{r}_1 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{r}) \mathscr{Z}(g, r)$$

which by a trivial Gaussian integration gives

$$\mathscr{Z}_{\text{path}}(g, \boldsymbol{r}_1) = \frac{1}{\cosh(gS/2)} \exp[-\tfrac{1}{4} g r_1^2 \tanh(gS/2)]. \tag{2.31}$$

Equations (2.28), (2.29) and (2.30) correspond, respectively, to Lévy's results (1.11), (1.12) and (1.14) in the introduction.

2.5. *Probabilities*

Let us calculate the probability distributions of the area of a Brownian ring and of the area between the arc and the chord of an open path. We have from (2.29)

$$P(A) \equiv \int_{-\infty}^{+\infty} \frac{\mathrm{d}g}{2\pi} \exp(\mathrm{i}gA) \frac{\mathscr{Z}(g)}{\mathscr{Z}(0)} = \frac{2}{S} \int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{2\pi} \exp(\mathrm{i}\xi u) \frac{u}{\sinh u} \tag{2.32}$$

with the reduced dimensionless variables

$$u = gS/2 \qquad \xi = 2A/S. \tag{2.33}$$

Integral (2.32) is easily done by closing the contour in the upper half-plane (for $\xi > 0$) to get a sum over the residues

$$\begin{aligned} P(A) &= \frac{2\pi}{S} \sum_{n \in \mathbb{N}} (-1)^{n+1} \exp(-n\pi\xi) n \\ &= \frac{2\pi}{S} \frac{\partial}{\partial \pi\xi} [1 + \exp(-\pi\xi)]^{-1}. \end{aligned} \tag{2.34}$$

This is

$$P(A) = \frac{\pi}{2S \cosh^2(\pi A/S)} \tag{2.35}$$

as announced in (1.5), valid for a closed Brownian curve.

For the area enclosed between a Brownian arc and its chord, we use (2.30) to get

$$\begin{aligned} P_{\text{chord}}(A) &\equiv \int_{-\infty}^{+\infty} \frac{\mathrm{d}g}{2\pi} \exp(\mathrm{i}gA) \mathscr{Z}_{\text{chord}}(g) \\ &= \frac{2}{S} \int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{2\pi} \exp(\mathrm{i}\xi u) \frac{1}{\cosh u}. \end{aligned} \tag{2.36a}$$

Closing the contour yields the series (for example $\xi > 0$)

$$\begin{aligned} P_{\text{chord}}(A) &= \frac{2}{S} \sum_{n \in \mathbb{N}} \exp(-\pi\xi/2)(-1)^n \exp(-n\pi\xi) \\ &= \frac{1}{S \cosh(\pi A/S)} \end{aligned} \tag{2.36b}$$

a probability which is symmetric with respect to positive and negative areas and normalised to 1.

It is worth remarking that Fourier inverting the generating functions with a constraint on the end-to-end distance r of the Brownian motion is not trivial. Examples are the arc-chord area generating function $\mathscr{Z}(g, r)$ (1.6) and (2.28), or the generating function $\mathscr{Z}_{\text{path}}(g, r)$ for the total arc-angle area (figure 2(*b*)). For instance for the arc-chord area, (1.6) leads to an integral

$$P(A, r)=\frac{2}{S}\int_{\mathbb{R}}\frac{\mathrm{d}u}{2\pi}\exp(\mathrm{i}\xi u)\frac{u}{\sinh u}\exp[(1-u\coth u)r^2/2S] \tag{2.37}$$

to which the residue theorem can no longer be applied, in contrast to integral (2.32) for $r\equiv 0$.

3. Area swept with respect to the centre of gravity

In this section we consider the area $\mathscr{A}_{\mathrm{G}}$ enclosed between the arc of an open planar Brownian motion and the angle joining its centre of mass G to its two extremities (figure 3). We can fix the centre of gravity at the origin of coordinates

$$\int_0^S \boldsymbol{r}(s)\,\mathrm{d}s=\mathbf{0} \tag{3.1}$$

and define a constrained generating function

$$\mathscr{Z}_{\mathrm{G}}(g)=\left\langle\exp(\mathrm{i}g\mathscr{A}_{\mathrm{G}}\{\boldsymbol{r}\})\delta^2\left(\int_0^S \boldsymbol{r}(s)\frac{\mathrm{d}s}{S}\right)\right\rangle. \tag{3.2}$$

Since G is at the origin the area $\mathscr{A}_{\mathrm{G}}$ can be written in the same way as in (2.4) and (2.5) and the explicit functional integral form of $\mathscr{Z}_{\mathrm{G}}$ is quite similar to (2.6):

$$\mathscr{Z}_{\mathrm{G}}(g)=\int \mathrm{d}\{\boldsymbol{r}\}\exp\left(-\tfrac{1}{2}\int \dot{\boldsymbol{r}}\cdot\mathbb{A}\cdot\dot{\boldsymbol{r}}\,\mathrm{d}s\,\mathrm{d}s'\right)\exp(\mathrm{i}g\,\tfrac{1}{2}\boldsymbol{r}(0)\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{r}(S))\delta^2\left(\int_0^S \boldsymbol{r}(s)\frac{\mathrm{d}s}{S}\right) \tag{3.3}$$

where $\mathbb{A}$ is the symmetric operator (2.7). In the functional integral (3.3) we integrate freely over the extremities $\boldsymbol{r}(0)$, $\boldsymbol{r}(S)$ of the path, and have simply to take into account

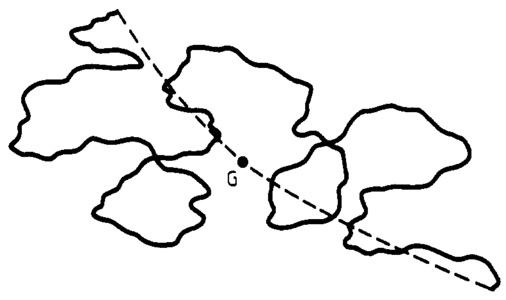

Figure 3. Area $\mathscr{A}_{\mathrm{G}}$ spanned by the Brownian arc with respect to the centre of gravity.

the condition (3.1). As in § 2 we shift to the functional variable $\dot{\boldsymbol{r}}(s)$ and use

$$\mathrm{d}\{\boldsymbol{r}\} = \mathrm{d}^2 r(0)\, \mathrm{d}\{\dot{\boldsymbol{r}}\}$$

$$\boldsymbol{r}(S) = \boldsymbol{r}(0) + \int_0^S \dot{\boldsymbol{r}}(s)\, \mathrm{d}s$$

$$\int_0^S \boldsymbol{r}(s)\, \mathrm{d}s/S = \boldsymbol{r}(0) + \int_0^S (S-s')\dot{\boldsymbol{r}}(s')\, \mathrm{d}s'/S.$$

The distribution in (3.3) is represented by a Fourier integral and one gets

$$\mathscr{Z}_{\mathrm{G}}(g) = \int \mathrm{d}\{\dot{\boldsymbol{r}}\} \exp\left(-\frac{1}{2}\int \dot{\boldsymbol{r}}\cdot \mathbb{A}\cdot \dot{\boldsymbol{r}}\, \mathrm{d}s\, \mathrm{d}s'\right) \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \exp\left(\mathrm{i}\boldsymbol{k}\cdot \int_0^S (S-s')\dot{\boldsymbol{r}}\frac{\mathrm{d}s'}{S}\right)$$
$$\times \int_{\mathbb{R}^2} \mathrm{d}^2 \boldsymbol{r}(0) \exp\left[\mathrm{i}\boldsymbol{r}(0)\cdot\left(\boldsymbol{k} + g\frac{1}{2}\boldsymbol{\varepsilon}\cdot \int_0^S \dot{\boldsymbol{r}}\, \mathrm{d}s\right)\right].$$

The integrals over $\boldsymbol{r}(0)$ and $\boldsymbol{k}$ are trivial to perform and one is left with a simple functional integral over $\dot{\boldsymbol{r}}$

$$\mathscr{Z}_{\mathrm{G}}(g) = \int \mathrm{d}\{\dot{\boldsymbol{r}}\} \exp\left(-\frac{1}{2}\int \dot{\boldsymbol{r}}\cdot \mathbb{A}\cdot \dot{\boldsymbol{r}}\, \mathrm{d}s\, \mathrm{d}s'\right)$$
$$\times \exp\left(-\mathrm{i}g\frac{1}{2S}\int_0^S\int_0^S \mathrm{d}s\, \mathrm{d}s'(S-s')\dot{\boldsymbol{r}}(s')\cdot \boldsymbol{\varepsilon}\cdot \dot{\boldsymbol{r}}(s)\right). \tag{3.4}$$

This integral is purely *quadratic* in $\boldsymbol{r}$ and we write it as

$$\mathscr{Z}_{\mathrm{G}}(g) = \int \mathrm{d}\{\dot{\boldsymbol{r}}\} \exp\left(-\frac{1}{2}\int \dot{\boldsymbol{r}}\cdot \mathbb{A}_{\mathrm{G}}\cdot \dot{\boldsymbol{r}}\, \mathrm{d}s\, \mathrm{d}s'\right) \tag{3.5}$$

with the *symmetric* Gaussian operator (see (2.7))

$$\mathbb{A}_{\mathrm{G}}(s, s') = \delta(s-s')\mathbf{1} + \mathrm{i}g\tfrac{1}{2}[\theta(s-s') - \theta(s'-s) - (s-s')/S]\boldsymbol{\varepsilon}. \tag{3.6}$$

$\mathbb{A}_{\mathrm{G}}$ is quite similar to operator (2.7), augmented by a new $s-s'$ term due to the centre of gravity constraint. The functional integral (3.5) is then formally (up to a proportionality constant) (see (2.10))

$$\mathscr{Z}_{\mathrm{G}}(g) = (\det \mathbb{A}_{\mathrm{G}})^{-1/2}. \tag{3.7}$$

We use the same method for calculating $\det \mathbb{A}_{\mathrm{G}}$ as in § 2.3. We define a function over interval $[-S, S]$:

$$f(s) \equiv \theta(s) - \theta(-s) - s/S. \tag{3.8}$$

We have to compute (see (3.6))

$$\ln \det \mathbb{A}_{\mathrm{G}} = \mathrm{Tr} \ln(\delta(s-s')\mathbf{1} + \tfrac{1}{2}\mathrm{i}g\boldsymbol{\varepsilon} f(s-s'))$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{\mathrm{i}g}{2}\right)^n \mathrm{Tr}\, \boldsymbol{\varepsilon}^n\, \mathrm{Tr} f^n \tag{3.9}$$

where

$$\mathrm{Tr} f^n \equiv \int_0^S f(s_1 - s_2) f(s_2 - s_3)\ldots f(s_n - s_1)\, \mathrm{d}s_1 \ldots \mathrm{d}s_n. \tag{3.10}$$

A useful identity is

$$\mathrm{Tr}\, \boldsymbol{\varepsilon}^n = 2\delta_{n,2\mathbb{Z}}(-1)^{n/2}. \tag{3.11}$$

For computing the trace (3.10) we cannot use the Fourier mode decomposition of $f(s)$ over $[-S, S]$ since the trace (3.10) is integrated over the half-period $[0, S]$ only. Hence, we have to diagonalise the integral operator associated with f, and search for the eigenvalues λ and eigenfunctions g such that

$$\int_0^S f(s-s')g(s')\,\mathrm{d}s' = \lambda g(s') \tag{3.12a}$$

where

$$f(s-s') = \theta(s-s') - \theta(s'-s) - (s-s')/S \tag{3.12b}$$

is *antisymmetric*. The explicit equation is

$$\int_0^S g(s')\,\mathrm{d}s' - \int_s^S g(s')\,\mathrm{d}s' + as + b = \lambda g(s) \tag{3.13}$$

with

$$a \equiv -\frac{1}{S}\int_0^S g(s')\,\mathrm{d}s' \qquad b \equiv \frac{1}{S}\int_0^S s'g(s')\,\mathrm{d}s'. \tag{3.14}$$

Differentiating (3.13) with respect to s yields

$$2g(s) + a = \lambda g'(s')$$

hence

$$g(s) = -\tfrac{1}{2}a + \mu \exp[(2/\lambda)s] \tag{3.15}$$

where μ is a free parameter fixing the L^2-norm of g. Since we differentiated (3.13), the condition that (3.13) holds at one point at least (e.g. $s=0$) has to be implemented

$$-\int_0^S g(s')\,\mathrm{d}s' + b = \lambda g(0)$$

i.e.

$$b + aS = \lambda(\mu - \tfrac{1}{2}a). \tag{3.16}$$

It is not difficult then to use solution (3.15) in (3.14), and finally substitute a and b in (3.16) to find the simple eigenvalue equation

$$\tanh(S/\lambda) = -\tfrac{1}{3}(S/\lambda) \tag{3.17}$$

whose solutions are pure imaginary, as expected from the antisymmetry of (3.12b). We set for the eigenvalues λ_m

$$S/\lambda_m = \mathrm{i}x_m \tag{3.18}$$

$$\begin{aligned} &\tan x_m = -\tfrac{1}{3}x_m \qquad x_m \in \left]-\tfrac{1}{2}\pi + m\pi, \tfrac{1}{2}\pi + m\pi\right[\qquad m \in \mathbb{Z}^* \\ &x_{-m} = -x_m \end{aligned} \tag{3.19}$$

where we have to discard the trivial zero $x_0 = 0$. In terms of the eigenvalues λ_m we now have for the convolution trace (3.10)

$$\mathrm{Tr}\, f^n = \sum_{m\in\mathbb{Z}^*} \lambda_m^n. \tag{3.20}$$

Inserting this and (3.11) into (3.9) gives immediately

$$\ln \det \mathbb{A}_G = - \sum_{m\in\mathbb{Z}^*} \sum_{n'=1}^{\infty} \frac{1}{n'} \left(\frac{g\lambda_m}{2}\right)^{2n'}$$

$$= \sum_{m\in\mathbb{Z}} \ln \left[1+\left(\frac{gS}{2x_m}\right)^2\right].$$

Hence the generating function (3.7) is given by the infinite product representation

$$\mathscr{Z}_G = (\det \mathbb{A}_G)^{-1/2} = \prod_{m\geq 1} \left[1+\left(\frac{gS}{2x_m}\right)^2\right]^{-1}. \tag{3.21}$$

One can calculate this infinite product as

$$\prod_{m\geq 1} \left(1+\frac{x^2}{x_m^2}\right) = \frac{1}{4}\left(\cosh x + 3\frac{\sinh x}{x}\right) \equiv \phi(x) \tag{3.22}$$

where the x_m are the solutions of (3.19). This identity can be obtained from the contour integral in the complex plane

$$I = \int_{\gamma_1 \cup \gamma_2} \ln\left(1-\frac{x^2}{z^2}\right) \frac{\phi'(z)}{\phi(z)} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \tag{3.23}$$

where γ_1 and γ_2 are two contours in the upper and lower half-plane, γ_1 encircling the poles $\mathrm{i}x_m$, $m \geq 1$ and γ_2 the others $-\mathrm{i}x_m = \mathrm{i}x_{-m}$, $m \geq 1$, zeros of $\phi(z)$, $\phi(\mathrm{i}x_m) = 0$ (figure 4). A trivial use of the residue theorem gives on the one hand

$$I = \sum_{m\in\mathbb{Z}^*} \ln\left(1+\frac{x^2}{x_m^2}\right). \tag{3.24}$$

On the other hand, one can deform the contour $\gamma_1 \cup \gamma_2$ in order to enclose only the cut $z \in \mathbb{R}$, $z \in [-x, x]$ of $\ln[1-(x^2/z^2)]$ (figure 4). A standard integration along this cut then gives I (3.23):

$$I = \ln[\phi(x)\phi(-x)/\phi^2(0)]. \tag{3.25}$$

From this follows the result (3.22); QED.

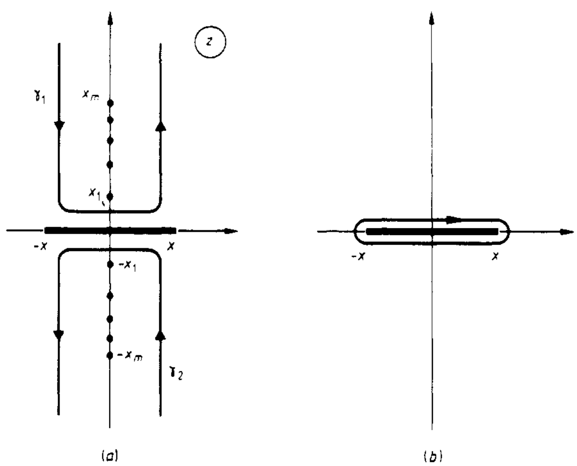

Figure 4. (*a*) The integral contour $\gamma_1 \cup \gamma_2$ for evaluating (3.23). (*b*) The deformed contour along the cut $[-x, x]$ of $\ln[1-(x^2/z^2)]$.

Finally we find for the $\mathscr{A}_G$-generating function (3.21)

$$\mathscr{Z}_G(g) = \frac{4}{\cosh u + 3\sinh u/u} \qquad u = gS/2 \tag{3.26}$$

which is a combination of the generating functions (2.29) $u/\sinh u$ for the area of a closed path, and (2.30) $1/\cosh u$ for the arc-chord area. This characteristic function (3.26) can be also derived (Yor 1989) from the results of Biane and Yor (1986). The probability distribution associated with the area $\mathscr{A}_G$ is now explicitly

$$\begin{aligned} P_G(A) &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}g}{2\pi} \exp(\mathrm{i}gA)\mathscr{Z}_G(g) \\ &= \frac{2}{S}\int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{2\pi} \exp(\mathrm{i}\xi u) \frac{4}{\cosh u + 3\sinh u/u} \end{aligned} \tag{3.27}$$

with $\xi = 2A/S$. By integration in the complex plane, one picks up the poles (3.19), zeros of $\phi(u)$ (3.22), and gets the series

$$P_G(A) = \frac{2}{S} \sum_{m\geq 1} \frac{x_m}{1+(x_m^2/12)} \frac{(-1)^{m+1}}{|\cos x_m|} \exp(-|\xi| x_m) \tag{3.28}$$

where the sum runs only over the positive roots x_m of (3.19). Asymptotically, the large deviations from the mean $\langle \mathscr{A}_G \rangle = 0$ are driven by the first pole

$$\underset{A\to\pm\infty}{P_G(A)} \simeq \frac{2}{S} \frac{x_1}{1+(x_1^2/12)} \frac{1}{|\cos x_1|} \exp[-(2/S)|A| x_1].$$

In conclusion, let us recall the principal probability distributions derived here: area enclosed by a Brownian ring (2.35); area enclosed between the Brownian arc and its chord (2.36); and finally area spanned between the Brownian arc and the wedge angle joining the centre of gravity to the extremities of the arc ((3.26) and (3.28)). The functional integral technique used to obtain (2.12) in § 2.1 was straightforward. The actual computation in §§ 2.2 and 2.3 of the inverse Gaussian operator and of the determinant by perturbation about the identity is of generic applicability (also in other functional integral problems). Any quadratic form associated with the Brownian motion could be studied in the same way.

Let us also mention that generating functions entirely similar to $\mathscr{Z}(g, \boldsymbol{r})$ (equation (1.11)) appear in the different problem of the winding angle distribution of a Brownian motion for some deep reasons related to the Ray-Knight theorem (1963) (see the discussion in Pitman and Yor (1986)). Finally, the area problem for a closed Brownian curve addressed here is also relevant to the statistical mechanics of two-dimensional (Brownian) vesicles (Leibler *et al* 1987). However, the introduction of an internal pressure p coupled to the area by $\exp(-p\mathscr{A})$, leads to a partition function

$$\mathbb{Z}(p) \equiv \langle \exp(-p\mathscr{A}) \rangle_{\text{ring}} = \frac{\mathscr{Z}(g=\mathrm{i}p)}{\mathscr{Z}(0)} = \frac{pS/2}{\sin pS/2}$$

and the poles correspond to the unstability generated by the possibility of both signs for the algebraic area.

Let us conclude with a (non-exhaustive) list of related mathematical works and generalisations of Lévy's stochastic area, which could be useful to the community of physicists: earlier calculation of $\langle \exp(-\lambda \int_0^S x^2(s)\, \mathrm{d}s) \rangle$ by Cameron and Martin (1945); heat equation by Gaveau (1977); iterated logarithm laws by Berthuet (1979, 1981) and

Helmes (1986); probabilistic proof of the index theorem (Bismut 1984, 1988); relations to Bessel processes (Pitman and Yor 1982a, 1982b, Biane and Yor 1986, 1987). Also noticeable are various generalisations to n-dimensional Brownian motions of the planar area process considered here, by Helmes and Schwane (1983) and Berthuet (1986). Krée (1986) derived and generalised to arbitrary dimensions the characteristic function of Lévy's stochastic area in terms of functional determinants, which bear some resemblance to ours. Recently, MacAonghusa and Pulé (1988) obtained an extension of Lévy's formula in which the two-dimensional Brownian motion is also conditioned by the value of the integral of its components with respect to some measures on $[0, S]$.

The results (3.26) and (3.28) concerning the probability distribution of the area swept by a planar Brownian motion with respect to its mean position (i.e. centre of gravity) could also find an application in Malliavin's calculus (Yor 1988).

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